A Discrete Time Valuation of Callable Financial Securities with Regime Switches

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Abstract In this paper, we consider a model of valuing callable financial securities when the underlying asset price dynamic is modeled by a regime switching process. The callable securities enable both an issuer and an investor to exercise their rights to call. We show that such a model can be formulated as a coupled stochastic game for the optimal stopping problem with two stopping boundaries. We provide analytical results of optimal stopping rules of the issuer and the investor under general payoff functions defined on the underlying asset price, the state of the economy and the time. In particular, we derive specific stopping boundaries for the both players by specifying for the callable securities to be the callable American call and put options. Also, numerical examples are presented to investigate the impact of parameters on the value function as well as on the optimal stopping rules.

Keywords: Optimal stopping; Game option; Markov chain; Regime switching; Callable securities; Stopping boundaries

1. Introduction
The purpose of this paper is to develop a dynamic valuation framework for callable financial securities with general payoff function by explicitly incorporating the use of regime switches. Such examples of the callable financial security may include game options (Kifer 2000, Kyprianou 2004), convertible bonds (Yagi and Sawaki 2005, 2007), callable put and call options (Black and Scholes 1973, Brennan and Schwartz 1976, Geske and Johnson 1984, McKean 1965). Most studies on these securities have focused on the pricing of the derivatives when underlying asset price processes follow a Brownian motion defined on a single probability space. In other words the realizations of the price process come from the same source of the uncertainty over the planning horizon.

The Markov regime switching model makes it possible to capture the structural changes of the underlying asset prices based on the macro-economic environment, fundamentals of the real economy and financial policies including international monetary cooperation. Such regime switching can be presented by the transition of the states of the economy, which follows a Markov chain. Recently, there is a growing interest in the regime switching model. Naik (1993), Guo (2001), Elliott et al. (2005) address the European call option price formula. Guo and Zhang (2004) presents a valuation model for perpetual American put options. Le and Wang (2010) study the optimal stopping time for the finite time horizon, and derive the optimal stopping strategy and properties of the solution. They also derive the technique for computing the solution and show some numerical examples for the American put option.

In this paper we show that there exists a pair of optimal stopping rules for the issuer and the investor and derive the value of the coupled game. Should the payoff functions be specified like options, some analytical properties of the optimal stopping rules and their values can be explored under several assumptions. In particular, we are interested in the cases of callable American put and call options in which we derive the optimal stopping boundaries of the both of the issuer and
the investor, depending on the state of the economy. Numerical examples are also presented to illustrate these properties.

The organization of our paper is as follows: In section 2, we formulate a discrete time valuation model for a callable contingent claim whose payoff functions are in general form. And then we derive optimal policies and investigate their analytical properties by using contraction mappings. Section 3 discusses two special cases of the payoff functions to derive the specific stop and continue regions for callable put and call, respectively. In Section 4 we present numerical results for the American callable put option using binomial model. Finally, last section concludes the paper with further comments. It summarize results of this paper and raises further directions for future research.

2. A Genetic Model of Callable-Putable Financial Commodities

In this section we formulate the valuation of callable securities as an optimal stopping problem in discrete time. Let \( \{1, 2, \cdots , N\} \) be the set of states of the economy and \( i \) or \( j \) denote one of these states. We denote \( Z_t \) be the finite Markov chain with transition probability \( P_{ij} = \Pr\{Z_{t+1} = j \mid Z_t = i\} \) which present the state of the economy at time \( t \). A transition from \( i \) to \( j \) means a regime switch. Let \( S_t \) be the asset price at time \( t \) and suppose

\[
S_{t+1} = S_t X_t^{Z_{t+1}} = S_0 X_1^Z X_2^Z \cdots X_{t+1}^Z
\]

provided that the states of the economy from time 1 through \( t + 1 \), \( (Z_1, Z_2, \cdots , Z_{t+1}) \) are observed, where \( X_t^i \) are independent positive random variables having mean \( \mu_i \) with the probability distribution \( F_i(\cdot) \). The sequences \( \{X_t^i\} \) and \( \{Z_t\} \) are assumed to be independent.

A callable contingent claim is a contract between an issuer I and an investor II addressing the asset with a maturity \( T \). The issuer can choose a stopping time \( \sigma \) to call back the claim with the payoff function \( f_\sigma \) and the investor can also choose a stopping time \( \tau \) to exercise his/her right with the payoff function \( g_\tau \) at any time before the maturity. Should neither of them stop before the maturity, the payoff is \( h_T \). The payoff always goes from the issuer to the investor. Here, we assume

\[
0 \leq g_t \leq h_t \leq f_t, \; 0 \leq t < T
\]

and

\[
g_T = h_T. \tag{2.2}
\]

The investor wishes to exercise the right to maximize the expected payoff. On the other hand, the issuer wants to call the contract to minimize the payment to the investor. Then, for any pair of the stopping times \( (\sigma, \tau) \), define the payoff function by

\[
R(\sigma, \tau) = f_\sigma 1_{\{\sigma < \tau \leq T\}} + g_\tau 1_{\{\tau < \sigma \leq T\}} + h_T 1_{\{\sigma \wedge \tau = T\}}. \tag{2.3}
\]

When the initial asset price \( S_0 = s \), our stopping problem becomes the valuation of

\[
v_t(s, i) = \min_{\sigma \in \mathcal{J}_t, \tau \in \mathcal{J}_t} \max_{\sigma \leq \tau} E_{s,i}[\beta^{\sigma \wedge \tau} R(\sigma, \tau)], \tag{2.4}
\]

where \( \beta, 0 < \beta < 1 \) is the discount factor, and \( \mathcal{J} \) is the finite set of stopping times taking values in \( \{t, t + 1, \cdots , T\} \). Since the asset price process follows a random walk, the payoff processes of \( g_t \) and \( f_t \) are both Markov types. We consider this optimal stopping problem as a Markov decision process. Let \( v_n(s, i) \) be the price of the callable CC when the asset price is \( s \) and the state is \( i \). Here, the trading period moves backward in time indexed by \( n = 0, 1, 2, \cdots , T \). It is easy to see that \( v_n(s, i) \) satisfies

\[
v_{n+1}(s, i) \equiv (uv_n)(s, i)
\]

\[
\equiv \min \left\{ f_{n+1}(s, i), \max \left( g_{n+1}(s, i), \beta \sum_{j=1}^{N} P_{ij} \int_0^\infty v_n(sx, j) dF_i(x) \right) \right\} \tag{2.5}
\]
with the boundary conditions are \( v_0(s,i) = h_0(s,i) \) for any \( s, i \) and \( v_n(s,0) \equiv 0 \) for any \( n \) and \( s \). Define the operator \( \mathcal{A} \) as follows:

\[
(\mathcal{A}v_n)(s,i) \equiv \beta \sum_{j=1}^{N} P_{ij} \int_{0}^{\infty} v_n(sx,j) dF_i(x).
\] (2.6)

**Remark 2.1.** The equation (2.5) can be reduced to the non-switching model when we set \( P_{ii} = 1 \) for all \( i \), or \( f_n(s,i) = f_n(s), g_n(s,i) = g_n(s), h_0(s,i) = h_0(s) \) and \( \mu_i = \mu \) for all \( i, n \) and \( s \).

Let \( V \) be the set of all bounded measurable functions with the norm \( \|v\| = \sup_{s \in [0,\infty)} |v(s,i)| \) for any \( i, v \in \mathbb{V} \). For \( u, v \in V \), we write \( u \leq v \) if \( u(s,i) \leq v(s,i) \) for all \( s \in (0,\infty) \). A mapping \( \mathcal{U} \) is called a contraction mapping if

\[
\|\mathcal{U}u - \mathcal{U}v\| \leq \beta\|u - v\|
\]

for some \( \beta < 1 \) and for all \( u, v \in V \).

**Lemma 2.1.** For any \( 0 < c_2 < c_1 \) and \( 0 < b \leq a \), we have

\[
\min\{a, \max(b, c_1)\} - \min\{a, \max(b, c_2)\} \leq \min(a, c_1) - \max(b, c_2).
\] (2.7)

**Lemma 2.2.** The mapping \( \mathcal{U} \) as defined by equation (2.5) is a contraction mapping.

**Corollary 2.1.** There exists a unique function \( v \in V \) such that

\[
(\mathcal{U}v)(s,i) = v(s,i) \quad \text{for all } s, i.
\] (2.8)

Furthermore, for all \( u \in V \),

\[
(\mathcal{U}^Tu)(s,i) \to v(s,i) \text{ as } T \to \infty,
\]

where \( v(s,i) \) is equal to the fixed point defined by equation (2.8), that is, \( v(s,i) \) is a unique solution to

\[
v(s,i) = \min\{f(s,i), \max(g(s,i), \mathcal{A}v)\}.
\]

Since \( \mathcal{U} \) is a contraction mapping from Corollary 2.1, the optimal value function \( v \) for the perpetual contingent claim can be obtained as the limit by successively applying an operator \( \mathcal{U} \) to any initial value function \( v \) for a finite lived contingent claim.

To establish an optimal policy, we make some assumptions;

**Assumption 2.1.**

(i) \( f_n(s,i) \geq f_n(s,j), g_n(s,i) \geq g_n(s,j) \) and \( h_n(s,i) \geq h_n(s,j) \) for each \( n \) and \( s \), and states \( i, j \), \( 1 \leq j < i \leq N \).

(ii) \( f_n(s,i), g_n(s,i) \) and \( h_n(s,i) \) are monotone in \( s \) for each \( i \) and \( n \), and are non-decreasing in \( n \) for each \( s \) and \( i \).

(iii) \( F_1(x) \geq F_2(x) \geq \cdots \geq F_N(x) \) for all \( x \).

(iv) For each \( k \), \( \sum_{j=k}^{N} P_{ij} \) is non-decreasing in \( i \).

**Lemma 2.3.** Suppose Assumption 2.1 holds.

(i) For each \( i \), \( (\mathcal{U}^n v)(s,i) \) is monotone in \( s \) for \( v \in V \).

(ii) \( v \) satisfying \( v = \mathcal{U}v \) is monotone in \( s \).

(iii) Suppose \( v_n(s,i) \) is monotone non-decreasing in \( s \), then \( v_n(s,i) \) is non-decreasing in \( i \).

(iv) \( v_n(s,i) \) is non-decreasing in \( n \) for each \( s \) and \( i \).

(v) For each \( i \), there exists a pair \( (s_n^{**}(i), s_n^{*}(i)) \), \( s_n^{**}(i) < s_n^{*}(i) \), of the optimal boundaries such that

\[
v_n(s,i) \equiv (\mathcal{U}v_{n-1})(s) = \begin{cases} f_n(s,i), & \text{if } s_n^{*}(i) \leq s, \\ \mathcal{A}v_{n-1}, & \text{if } s_n^{**}(i) < s < s_n^{*}(i), n = 1, 2, \ldots, T, \\ g_n(s,i), & \text{if } s \leq s_n^{**}(i), \end{cases}
\]

with \( v_0(s,i) = h_0(s,i) \).
Proof.

(i) The proof follows by induction on $n$. For $n = 1$, we have

$$(\mathcal{U}^1v)(s, i) = \min \left\{ f_1(s, i), \max \left( g_1(s, i), \beta \sum_{j=1}^{N} P_{ij} \int_{0}^{\infty} h_0(sx, j) dF_i(x) \right) \right\}$$

which, since Assumption 2.1 (ii), implies that $(\mathcal{U}^1v)(s, i)$ is monotone in $s$. Suppose that $(\mathcal{U}^nv)(s, i)$ is monotone for $n > 1$. Then, we have

$$(\mathcal{U}^{n+1}v)(s, i) = \min \left\{ f_{n+1}(s, i), \max \left( g_{n+1}(s, i), \beta \sum_{i=1}^{n} P_{ij} \int_{0}^{\infty} (\mathcal{U}^nv)(sx, j) dF_i(x) \right) \right\},$$

which is again monotone in $s$.

(ii) Since $\lim_{n \to \infty} (\mathcal{U}^nv)(s, i)$ point-wisely converges to the limit $v(s, i)$ from corollary 2.1, the limit function $v(s, i)$ is also monotone in $s$.

(iii) For $n = 0$, it follows from Assumption 2.1 (i) that $v_0(s, i) = h_0(s, i)$ is non-decreasing in $i$. Suppose (iii) holds for $n$. If $v_n(s, i)$ is monotone non-decreasing in $s$, then $v_n(s, i)$ is also monotone non-decreasing in $x$ for each $s$. Then, from Assumption 2.1 (iii), we obtain

$$\beta \sum_{j=1}^{N} P_{ij} \int_{0}^{\infty} v_n(sx, j) dF_i(x) \leq \beta \sum_{j=1}^{N} P_{ij} \int_{0}^{\infty} v_n(sx, j) dF_{i+1}(x)$$

$$= \beta \int_{0}^{\infty} \sum_{k=1}^{N} (v_n(sx, k) - v_n(sx, k - 1)) \sum_{j=1}^{N} P_{ij} dF_{i+1}(x)$$

$$\leq \beta \int_{0}^{\infty} \sum_{k=1}^{N} (v_n(sx, k) - v_n(sx, k - 1)) \sum_{j=1}^{N} P_{i+1j} dF_{i+1}(x)$$

$$= \beta \sum_{j=1}^{N} P_{i+1j} \int_{0}^{\infty} v_n(sx, j) dF_{i+1}(x),$$

where the second inequality follows from Assumption 2.1 (iv). Hence, we obtain

$$v_{n+1}(s, i) = \min \{ f_{n+1}(s, i), \max( g_{n+1}(s, i), \mathcal{A}v_n(s, i) ) \}$$

$$\leq \min \{ f_{n+1}(s, i + 1), \max( g_{n+1}(s, i + 1), \mathcal{A}v_n(s, i + 1) ) \}$$

$$= v_{n+1}(s, i + 1).$$

(iv) For $n = 1$ in equation (2.5), it follows from Assumption 2.1 (ii) that

$$v_1(s, i) = \min \{ f_1(s, i), \max( g_1(s, i), \mathcal{A}v_0 ) \}$$

$$\geq \min \{ f_1(s, i), g_1(s, i) \} = g_1(s, i) \geq g_0(s, i) = v_0(s, i).$$

Suppose (iv) holds for $n$. We obtain

$$v_{n+1}(s, i) = \min \{ f_{n+1}(s, i), \max( g_{n+1}(s, i), \mathcal{A}v_n ) \}$$

$$\geq \min \{ f_n(s, i), \max( g_n(s, i), \mathcal{A}v_{n-1} ) \}$$

$$= v_n(s, i).$$

(iv) Should $v_n(s, i) = (\mathcal{U}^{n-1}v)(s, i)$ be monotone in $s$, then there exists at least one pair of boundary values $s_n^*(i)$ and $s_n^*(i)$ such that

$$v_n(s, i) = \begin{cases} f_n(s, i), & \text{if } s \geq s_n^*(i), \\ \max( g_n(s, i), \mathcal{A}v_{n-1} ), & \text{otherwise}, \end{cases}$$
and
\[
\max(g_n(s, i), \mathcal{A}v_{n-1}) = \begin{cases} 
\ g_n(s, i), & \text{for } s \leq s_n^{**}(i), \\
\mathcal{A}v_{n-1}, & \text{otherwise}.
\end{cases}
\]

**Corollary 2.2.** The relationship between \(g_n\), \(f_n\) and \(v_n(s, i)\) is given by
\[
g_n(s, i) \leq v_n(s, i) \leq f_n(s, i).
\]

**Proof.** The proof directly follows from equation (2.5).

We define the stopping regions \(S^I\) for the issuer and \(S^{II}\) for the investor as
\[
S^I_n(i) = \{ (s, n, i) \mid v_n(s, i) \geq f_n(s, i) \}, \quad (2.12)
S^{II}_n(i) = \{ (s, n, i) \mid v_n(s, i) \leq g_n(s, i) \}. \quad (2.13)
\]
Moreover, the optimal exercise boundaries for the issuer and the investor are defined as
\[
s^*_n(i) = \inf \{ s \in S_n^I(i) \}, \quad (2.14)
s^{**}_n(i) = \inf \{ s \in S_n^{II}(i) \}. \quad (2.15)
\]

### 3. A Simple Callable American Option with Regime Switching

Interesting results can be obtained for the special cases when the payoff functions are specified. In this section we consider callable American options whose payoff functions are specified as a special case of callable CC. If the issuer call back the claim in period \(n\), the issuer must pay to the investor \(g_n(s, i) + \delta^i_n\). Note that \(\delta^i_n\) is the compensating for the contract cancellation, and varies depending on the state and the time period. If the investor exercises his/her right at any time before the maturity, the investor receives the amount \(g_n(s, i)\). In the following subsections, we discuss the optimal cancel and exercise policies both for the issuer and investor and show the analytical properties under some conditions.

**3.1. Callable Call Option**

We consider the case of a callable call option where \(g_n(s, i) = (s - K^i)^+\) and \(f_n(s, i) = g_n(s, i) + \delta^i_n\), \(0 < \delta^i_n < K^i\). Here, \(K^i\) is the strike price on the state \(i\). The stopping regions for the issuer \(S^I_n(i)\) and investor \(S^{II}_n(i)\) with respect to the callable call option are given by
\[
\begin{cases}
S^I_n(i) = \{ s \mid v_n(s, i) \geq (s - K^i)^+ + \delta^i_n \}, & \text{for } n = 1, \cdots, T, \\
S^I_0(i) = \phi, & \text{for } n = 0, \\
S^{II}_n(i) = \{ s \mid v_n(s, i) \leq (s - K^i)^+ \}, & \text{for } n = 0, 1, \cdots, T.
\end{cases}
\]

For each \(i\) and \(n\), we define the thresholds for the callable call option as
\[
s^*_n(i) = \inf \{ s \mid v_n(s, i) = (s - K^i)^+ + \delta^i_n \},
\]
\[
s^{**}_n(i) = \inf \{ s \mid v_n(s, i) = (s - K^i)^+ \}.
\]

The following remark represents the well known result that American call options are identical to the corresponding European call options.

**Remark 3.1.** If the asset price \(S_t\) is a martingale, then callable call option with the maturity \(T < \infty\) can be degenerated into callable European, that is \(S^{II}_n(i) = \phi\) for \(n > 0\) and \(S^I_0(i) = \{ K^i \}\) for each \(i\).
In the case of callable-putable call claims, it follows that it is optimal for the investor not to exercise his/her putable right before the maturity. However, the issuer should choose an optimal call stopping time so as to minimize the expected payoff function:

\[ v_t(s, i) = \min_{\sigma \in \mathcal{B}_{i,T}} E^\sigma_0 [e^{\beta \sigma} f_t(S_\sigma, i)1_{\{\sigma < T\}} + \beta^T h_T 1_{\{\sigma = T\}}], \]

where \( E^\sigma \) is the expectation under the risk neutral measure.

We set out the assumptions to show the analytical properties of the optimal exercise policies.

**Assumption 3.1.**

(i) \( \beta \mu_N \leq 1 \)

(ii) \( K_1 \geq K_2 \geq \cdots \geq K_N \geq 0 \).

(iii) \( 0 \leq \delta_{n}^j \leq \delta_{n+1}^j \leq \cdots \leq \delta_{N}^j \) for each \( n \).

(iv) \( \delta_{n}^i > 0 \) and \( \delta_{n}^i \) is non-decreasing and concave in \( n > 0 \) for each \( i \).

(v) \( \beta \sum_{j=1}^{N} P_{n+1}^i K_j - K_{i+1} \leq \beta \sum_{j=1}^{N} P_{n}^j K_j - K_i \) for each \( i \).

**Remark 3.2.** For example, \( \delta_{n}^i = \delta_{n}^i e^{-r(T-n)} = \frac{\delta_{n}^i}{(1+r)^{T-n}} \) satisfies Assumption 3.1(iv).

**Lemma 3.1.** If Assumption 3.1 (i) holds, then \( v_n(s, i) - s \) is decreasing in \( s > K^i \), and \( v_n(s, i) \) is non-decreasing in \( s \) for \( s \leq K^i \) for each \( n, i \).

**Proof.** We prove it by induction. For \( n = 0 \), the claim certainly holds. It is sufficient to prove for the case of \( s > K^i \). Suppose the claim holds for \( n \), then we have

\[
\begin{align*}
v_{n+1}(s, i) - s &= \min \{s - K^i + \delta_{n+1}^i, \max(s - K^i, \mathcal{A}v_n)\} - s \\
&= \min \left\{-K^i + \delta_{n+1}^i, \max \left(-K^i, \beta \sum_{j=1}^{N} P_{n}^j \int_{0}^{\infty} (v_n(sx, j) - sx) dF_t(x) + (\beta \mu_i - 1)s \right) \right\}.
\end{align*}
\]

Since the statement is true for \( n \), \( v_n(sx, j) - sx \) is decreasing in \( s \) for \( x > K^i \). Assumption 2.1 (iii) implies that \( \mu_1 \leq \mu_2 \leq \cdots \leq \mu_N \). If \( \mu_N \leq \frac{1}{\beta} \), then \( (\beta \mu_i - 1)s \) is non-increasing in \( s \). Hence, \( v_{n+1}(s, i) - s \) is decreasing in \( s > K^i \).

**Lemma 3.2.** Suppose Assumption 3.1 (i)-(v) hold.

(i) \( v_n(s, i) \) is non-decreasing in \( i \) for each \( s \) and \( n \).

(ii) \( v_n(s, i) + K^i \) is non-decreasing in \( i \) for each \( s > K^i \) and \( n \).

**Proof.**

(i) By the induction, it is shown that \( v_n(s, i) \) is non-decreasing in \( s \). Hence, the proof follows directly from Lemma 2.3 (iii).

(ii) For \( K^i < s \), the statement (ii) certainly holds for \( n = 0 \). Suppose (ii) holds for \( n \). For \( n + 1 \), we have

\[
v_{n+1}(s, i) + K^i = \min \{s + \delta_{n+1}^i, \max(s, \mathcal{A}v_n + K^i)\}.
\]

Now we show that \( \mathcal{A}v_n(s, i) + K^i \leq \mathcal{A}v_n(s, i+1) + K^{i+1} \). By Lemma 3.1, Assumption 3.1 (ii) and
There exists a time period $n^*_i$ for each $i$ such that $n^*_i \equiv \inf \{n \mid \delta^*_n \leq v^a_n(K^i, i)\}$, where $v^a_n(s, i) = \max\{(s - K^i)^+, \mathcal{A}v_{n-1}(s, i)\}$. Moreover, if $n^*_i \leq n \leq T$, we have $S^I_n(i) = \{K^i\}$. If $0 \leq n < n^*_i$, we have $S^I_n(i) = \phi$.

(ii) $n^*_i$ is non-decreasing in $i$.

**Proof.**

(i) Let $w^a_n(K^i, i) \equiv v^a_n(K^i, i) - \delta^*_n$. For $n = 0$, we have $w^a_0(K^i, i) = v^a_0(K^i, i) - \delta^*_0 < 0$ by Assumption 3.1 (iv). Since $\delta^*_n$ is concave in $n$ and $v^a_n(K^i, i)$ is convex in $n$, the function $w^a_n(K^i, i)$ is convex in $n$. Hence, there exists a value $n^*_i$ such that $n^*_i \equiv \inf\{n \mid \delta^*_n \leq v^a_n(K^i, i)\}$.

Let $\Psi^I_n(s, i) = v_n(s, i) - (s - K^i)^+ - \delta^*_n$. When $s = K^i$ for $n^*_i \leq n \leq T$, we have

$$v_n(K^i, i) = \min\{0, \max\{0, \mathcal{A}v_{n-1}(K^i, i)\} - \delta^*_n\} + \delta^*_n = \delta^*_n.$$

Thus, we obtain $\Psi^I_n(K^i, i) = 0$ for each $i$. Since the function $\Psi^I_n(s, i)$ is non-decreasing for $s \leq K^i$ and is decreasing for $K^i < s$ by Lemma 3.1, it is unimodal function in $s$, and $K^i$ is a maximizer of $\Psi^I_n(s, i)$. Thus, $v_n(s, i) < (s - K^i)^+ + \delta^*_n$ if $s \neq K^i$. Therefore, $S^I_n(i) = \{K^i\}$ for $n^*_i \leq n \leq T$.

For $0 \leq n < n^*_i$, since $\delta^*_n > v^a_n(K^i, i)$, we have

$$v_n(K^i, i) = \min\{0, v^a_n(K^i, i) - \delta^*_n\} + \delta^*_n = v^a_n(K^i, i) < \delta^*_n \leq (s - K^i)^+ + \delta^*_n.$$

Hence, we have $\Psi^I_n(K^i, i) < 0$, so $S^I_n(i) = \phi$.

(ii) For $n = 0$, $v^a_0(K^i, i) - \delta^*_0 = -\delta^*_0$ is non-increasing in $i$. By induction, we can show that $v^a_n(K^i, i) - \delta^*_n$ is non-increasing in $i$. Thus, since $v^a_n(K^i, i) - \delta^*_n$ is non-decreasing in $n$, the value $n^*_i$ is non-decreasing in $i$.

**Lemma 3.4.** Suppose Assumption 3.1 (i) holds. Then, there exists an optimal exercise policy for the both players $s^*_n(i) < s^*_n(i)$ such that the investor exercises the option if $s^*_n(i) \leq s$ and the issuer exercises the option if $s \leq s^*_n(i)$.

**Proof.** We first consider the optimal exercise policy for the investor. Let $\Psi^{II}_n(s, i) \equiv v_n(s, i) - (s - K^i)^+$. The investor does not exercise the option when $s < K^i$ because he/she wishes to exercise the right so as to maximize the expected payoff. For $s \geq K^i$, $\Psi^{II}_n(s, i)$ is non-decreasing in $s$ by Lemma 3.1. Since $v_n(K^i, i) \geq 0$, there exists a value $s^*_n(i)$ such that $\Psi^{II}_n(s, i) = 0$. For $s^*_n(i) \leq s$, $v_n(s, i) \leq (s - K^i)^+$. Hence, it is optimal for the investor to exercise the option when $s^*_n(i) \leq s$.

It follows from Lemma 3.3 (i) that the optimal exercise policy for the issuer is $s^*_n(i) = K^i$ for $n^*_i \leq n \leq T$ and $s^*_n(i) = \infty$ for $0 \leq n < n^*_i$. Since $\Psi^{II}_n(s, i)$ is decreasing in $s$ for $K^i < s$, we have $s^*_n(i) < s^*_n(i)$ for each $i$ and $n \in [n^*_i, T]$.
Lemma 3.5. (i) \( s^{**}_n(i) \) is non-decreasing in \( i \) for each \( n \).
(ii) \( s^{**}_n(i) \) is non-decreasing in \( n \) for each \( i \).

Proof. It is sufficient to consider the case of \( s > K^i \).
(i) When \( K^1 < K^2 < \cdots < K^N \), by Lemma 3.1 and Lemma 3.2 (ii), \( v_n(s, i) - s \) is decreasing in \( s \) and \( v_n(s, i) + K^i \) is non-decreasing in \( i \). Hence, we have

\[
\begin{align*}
s^{**}_n(i) &= \inf \{ s \mid v_n(s, i) - s = -K^i \} \\
&\leq \inf \{ s \mid v_n(s, i + 1) - s = -K^{i+1} \} \\
&= s^{**}_n(i + 1).
\end{align*}
\]

When \( K^i = K \) for all \( i \), by Lemma 3.2 (ii), the similar arguments lead to the result.
(ii) Since Lemma 2.3 (iv), \( v_n(s, i) \) is non-decreasing in \( n \). Thus, we have

\[
\begin{align*}
s^{**}_n(i) &= \inf \{ s \mid v_n(s, i) - s = -K^i \} \\
&\leq \inf \{ s \mid v_{n+1}(s, i) - s = -K^i \} \\
&= s^{**}_{n+1}(i).
\end{align*}
\]

\( \square \)

Theorem 3.1. Suppose that Assumption 3.1 (i)-(v) holds. The stopping regions for the issuer and investor can be obtained as follows:

(i) The optimal stopping region for the issuer:

\[
\begin{align*}
S^I_n(i) &= \{ K^i \}, \quad \text{if } n^*_i \leq n \leq T, \\
S^I_n(i) &= \phi, \quad \text{if } 0 \leq n < n^*_i,
\end{align*}
\]

where \( K^1 \geq K^2 \geq \cdots \geq K^N \geq 0 \), and \( n^*_i \equiv \inf \{ n \mid \delta^i_n \leq v^a_n(K^i, i) \} \) which is non-decreasing in \( i \). Here, \( v^a_n(s, i) = \max \{ (s - K^i)^+, Av_{n-1}(s, i) \} \).

(ii) The optimal stopping region for the investor:

\[
\begin{align*}
S^I_n(i) &= [s^{**}_n(i), \infty), \quad \text{if } n > 0, \\
S^I_n(i) &= \{ K^i \}, \quad \text{if } n = 0,
\end{align*}
\]

where \( s^{**}_n(i) \) is non-decreasing in \( n \) and \( i \). Moreover, \( s^*_n(i) \leq s^{**}_n(i) \) for each \( i \) and \( n \).

Proof. Part (i) follows from Lemma 3.3. Part (ii) is obtained from Lemma 3.4 and 3.5. For \( n = 0 \), since \( s^{**}_n(i) = \inf \{ s \mid (s - K^i)^+ \leq s - K^i \} = K^i \), we obtain \( S^I_0(i) = \{ K^i \} \). \( \square \)

Corollary 3.1. If \( \beta \mu_1 > 1 \) and \( K^i > \beta \sum_{j=1}^N P_{ij} K^j \), then it is never optimal for the investor to exercise before the maturity. It is never optimal for the issuer to call at the maturity.

Proof. For \( n = 0 \), the stopping regions for the issuer and investor are given by \( S^I_0(i) = \phi \) and \( S^I_0(i) = \{ K^i \} \), respectively. For \( n = 1 \), we have

\[
\begin{align*}
v_1(s, i) &= \min \{ s - K^i + \delta^i_1, \max \{ s - K^i, Av_0 \} \} \\
&= s - K^i + \min \{ \delta^i_1, \max \{ 0, -s + K^i + Av_0 \} \}
\end{align*}
\]
Here, we obtain

\[-s + K^i + Av_0 = -s + K^i + \beta \sum_{j=1}^{N} P_{ij} \int_{0}^{\infty} (sx - K^j)^+ dF_i(x) dx\]

\[= -s + K^i + \beta \sum_{j=1}^{N} P_{ij} \left\{ s \left( \mu_i - \int_{0}^{K^j} x dF_i(x) \right) - K^j \left( 1 - F_i \left( \frac{K^j}{s} \right) \right) \right\} \geq s(\beta \mu_i - 1) + K^i - \beta \sum_{j=1}^{N} P_{ij} K^j > 0.\]

It implies that \( v_2(s, i) > s - K^i, \) so \( s^*_n(i) = \infty. \) Since, from Lemma 3.5, \( s^*_n(i) \) is non-decreasing in \( n. \) Hence, \( S_n^{II}(i) = \{ \phi \} \) for each \( i \) and \( n \geq 1. \)

### 3.2. Callable Put Option

We consider the case of a callable put option where \( g_n(s, i) = \max\{K^i - s\} \) and \( f_n(s, i) = g_n(s, i) + \delta_n^i. \) The stopping regions for the issuer \( S_n^I(i) \) and the investor \( S_n^{II}(i) \) with respect to the callable put option are given by

\[
\begin{align*}
S_n^I(i) &= \{ s | v_n(s, i) \geq (K^i - s)^+ + \delta_n^i \}, & \text{for } n = 1, \ldots, T, \\
S_n^I(i) &= \phi, & \text{for } n = 0, \\
S_n^{II}(i) &= \{ s | v_n(s, i) \leq (K^i - s)^+ \}, & \text{for } n = 0, 1, \ldots, T.
\end{align*}
\]

For each \( i \) and \( n, \) we define the optimal exercise boundaries for the issuer \( \tilde{s}_n^*(i) \) and the investor \( \tilde{s}^*_n(i) \) as

\[
\begin{align*}
\tilde{s}_n^*(i) &= \inf \{ s | v_n(s, i) = (K^i - s)^+ + \delta_n^i \}, \\
\tilde{s}^*_n(i) &= \inf \{ s | v_n(s, i) = (K^i - s)^+ \}.
\end{align*}
\]

### Assumption 3.2.

(i) \( \beta \mu_N \leq 1 \)

(ii) \( 0 \leq K^1 \leq K^2 \leq \cdots \leq K^N. \)

(iii) \( 0 \leq \delta_n^1 \leq \delta_n^2 \leq \cdots \leq \delta_n^N \) for each \( n. \)

(iv) \( \delta_n^{\upsilon} > 0 \) and \( \delta_n^{\upsilon} \) is non-decreasing and concave in \( n > 0 \) for each \( i. \)

(v) \( \beta \sum_{j=1}^{N} P_{i+1j} - K^i+1 \geq \beta \sum_{j=1}^{N} P_{ij} K^j - K^i \) for each \( i. \)

### Lemma 3.6. If Assumption 3.2 (i) holds, then \( v_n(s, i) + s \) is increasing in \( s \) for \( s < K^i, \) and \( v_n(s, i) \) is non-increasing in \( s \) for \( K^i \leq s. \)

**Proof.** It is sufficient to prove for the case of \( s < K^i. \) The claim holds for \( n = 0. \) Suppose the assertion holds for \( n. \) Then, we have

\[
v_{n+1}(s, i) + s = \min \{ K^i + s + \delta^i_{n+1}, \max(\{K^i - s, Av_n\}) \} + s = \min \left\{ K^i + \delta^i_{n+1}, \max \left( K^i, \beta \sum_{j=1}^{N} P_{ij} \int_{0}^{\infty} (v_n(sx, j) + sx) dF_i(x) + (1 - \beta \mu_i)s \right) \right\}.\]

Hence, from Assumption 3.2 (i), \( v_{n+1}(s, i) + s \) is increasing in \( s \) for \( s < K^i. \) \( \square \)
Lemma 3.7.
(i) $v_n(s, i)$ is non-decreasing in $i$ for each $s$ and $n$.
(ii) $v_n(s, i) - K^i$ is non-decreasing in $i$ for each $s < K^i$ and $n$.

Proof.
(i) Let $w_n(s, i) = v_n(s, i) + s$. For $n = 0$, the statements certainly hold. It follows from Lemma 3.6 that $w_n(s, i)$ is increasing in $s$ for $s < K^i$. On the other hand, for $s \geq K^i$, we can show that $w_n(s, i)$ is increasing in $s$ by induction. Suppose (i) holds for $n$, then we have

$$w_{n+1}(s, i) = \min\{(K^i - s) + \delta^i_{n+1}, \max((K^i - s)^+, Aw_n(s, i))\} \leq \min\{(K^i - s) + \delta^i_{n+1} + 1, \max((K^i - s)^+, Aw_n(s, i + 1))\} = w_{n+1}(s, i + 1).$$

Thus, we obtain $v_{n+1}(s, i) \leq v_{n+1}(s, i + 1)$.

(ii) When $n = 0$, the claim holds. For $K^i > s$, we set $w_n(s, i) \equiv v_n(s, i) + s$. Suppose (ii) holds for $n$. Then, we have

$$w_{n+1}(s, i) - K^i = \min\{K^i - s + \delta^i_{n+1}, \max(K^i - s, Aw_n(s, i))\} - K^i = \min\{-s + \delta^i_{n+1}, \max(-s, Aw_n(s, i) - K^i)\}$$

By Lemma 3.6, we have

$$Aw_n(s, i) = \beta \sum_{j=1}^{N} P_{ij} \int_{0}^{\infty} w_n(sx, j) dF_i(x) \leq \beta \sum_{j=1}^{N} P_{ij} \int_{0}^{\infty} w_n(sx, j) dF_{i+1}(x) \leq Aw_n(s, i + 1) - \beta \sum_{j=1}^{N} (P_{i+1j} - P_{ij}) K^j.$$

From Assumption 3.2 (v), we obtain $Aw_n(s, i) - K^i \leq Aw_n(s, i + 1) - K^i$, so $v_{n+1}(s, i) - K^i \leq v_n(s, i) - K^i$. Hence, $w_{n+1}(s, i) - K^i \leq w_n(s, i + 1) - K^i$, so $v_{n+1}(s, i) - K^i \leq v_n(s, i) - K^i$. □

Lemma 3.8.
(i) There exists a time $n^*_i$ for each $i$ such that $n^*_i \equiv \inf\{n \mid \delta^i_n \leq v_n^*(K^i, i)\}$, where $v_n^*(s, i) = \max\{(K^i - s)^+, Aw_{n-1}(s, i)\}$. Moreover, if $n^*_i \leq n \leq T$, we have $S^I_n(i) = \{K_i^I\}$. If $0 \leq n < n^*_i$, we have $S^I_n(i) = \phi$.

(ii) $n^*_i$ is non-decreasing in $i$.

Proof. The proof can be done similarly as in the case of the call option in Lemma 3.3. □

Lemma 3.9. Suppose Assumption 3.2 (i) holds. Then, there exists an optimal exercise policy for the both players, and $s^*_n(i) < \bar{s}^*_n(i)$ such that the investor exercises the option if $s \leq s^*_n(i)$ and the issuer exercises the option if $s^*_n(i) \leq s$.

Proof. The proof is similar to the proof in the Lemma 3.4. □

Lemma 3.10.
(i) $\bar{s}^*_n(i)$ is non-increasing in $i$ for each $n$.
(ii) $\bar{s}^*_n(i)$ is non-increasing in $n$ for each $i$.

Proof. We only consider the case of $K^i > s$.  

(i) By Lemma 3.6, \( v_n(s, i) + s \) is increasing in \( s \) for \( K^i > s \). Therefore, from Lemma 3.7 (i), we have

\[
\tilde{s}_n^{**}(i) = \inf\{ s \mid v_n(s, i) + s = K^i \} \\
\geq \inf\{ s \mid v_n(s, i + 1) + s = K^{i+1} \} \\
= \tilde{s}_n^{**}(i + 1).
\]

(ii) By Lemma 2.3 (iv), \( v_n(s, i) \) is non-increasing in \( n \), so we have

\[
\tilde{s}_n^{**}(i) = \inf\{ s \mid v_n(s, i) + s = K^i \} \\
\geq \inf\{ s \mid v_{n+1}(s, i) + s = K^{i+1} \} \\
= \tilde{s}_{n+1}^{**}(i).
\]

\[\square\]

**Theorem 3.2.** Suppose that Assumption 3.2 (i)-(v) holds. The stopping regions for the issuer and investor can be obtained as follows:

(i) The optimal stopping region for the issuer:

\[
\begin{cases}
S_{n}^I(i) = \{ K^i \}, & \text{if } n_i^* \leq n \leq T, \\
S_{n}^I(i) = \phi, & \text{if } 0 \leq n < n_i^*,
\end{cases}
\]

where \( 0 \leq K^1 \leq K^2 \leq \cdots \leq K^N \), and \( n_i^* \equiv \inf\{ n \mid \delta_n \leq v_n^0(K^i, i) \} \) which is non-decreasing in \( i \). Here, \( v_n^0(s, i) = \max\{ (K^i - s)^+, A \nu_{n-1}(s, i) \} \).

(ii) The optimal stopping region for the investor:

\[
\begin{cases}
S_{n}^{II}(i) = [0, \tilde{s}_n^{**}(i)], & \text{if } n > 0, \\
S_{n}^{II}(i) = \{ K^i \}, & \text{if } n = 0,
\end{cases}
\]

where \( \tilde{s}_n^{**}(i) \) is non-increasing in \( n \) and \( i \). Moreover, \( \tilde{s}_n^{**}(i) \leq \tilde{s}_n(i) \) for each \( i \) and \( n \).

**Proof.** Part (i) follows from Lemma 3.8. Part (ii) can be obtained by Lemma 3.9 and 3.10. For \( n = 0 \), since \( \tilde{s}_n^{**}(i) = \inf\{ s \mid (s - K^i)^+ \leq s - K^i \} = K^i \), we have \( S_0^{II}(i) = \{ K^i \} \). \[\square\]

4. **Numerical Examples**

In this section we provide a numerical example for a callable American option by using the binomial tree model. We consider the regime-switching volatility model in a two-state setting for volatility. We write the volatility as

\[
\sigma(t) = \begin{cases}
\sigma_1, & \text{(the low volatility state)}, \\
\sigma_2, & \text{(the high volatility state)}.
\end{cases}
\]

The transition probability matrix is assumed to

\[
P = \begin{pmatrix}
p_1 & 1 - p_1 \\
1 - p_2 & p_2
\end{pmatrix}.
\]

For a fixed \( T \), let us divide the interval \([0, T]\) into \( M \) subintervals such that \( T = hM \). We set \( u_i = e^{\sigma(t)\sqrt{h}}, d_i = e^{-\sigma(t)\sqrt{h}} \) and

\[
q_i = \frac{e^{h \beta} - d_i}{u_i - d_i}, \quad i = 1, 2.
\]
We consider the joint Markov process \((S_t, Z_t), 0 \leq t \leq T\). Let \((S_n, Z_n) := (S_t, Z_t)_{t=nh}\) be the state at the period \(n, n = 0, 1, \ldots, M\). Assume that \((S_n, Z_n) = (S, i)\), then the joint system is described by

\[
(S_{n+1}, Z_{n+1}) = \begin{cases} (S_{u,i}), & \text{w.p. } q_ip_i, \\ (S_{u,j}), & \text{w.p. } q_i(1-p_i), \\ (S_{d,i}), & \text{w.p. } (1-q_i)p_i, \\ (S_{d,j}), & \text{w.p. } (1-q_i)(1-p_i), \end{cases} \tag{4.4}
\]

where \(i = 1, 2, i \neq j\). It is easy to show that the process is a martingale. The asset price after \(n\) periods on tree can be obtained by

\[
S_n = S_0v_0^n a_1^n d_0^n d_1^n \tag{4.5}
\]

where \(n_1 + n_2 + n_3 + n_4 = n\).

**Remark 4.1.**

Aingworth et al. (2006) show that the number of distinct underlying prices at period \(n\) is \(n+2N-1\) or \(C_{2N-1}\). Here, \(N\) is the number of the state of the economy.

Let \(\bar{v}_n^i(n_1, n_2, n_3)\) be the value of the callable American put at time period \(n\) when the number of the up moves in the state 1 is \(n_1\), the number of the up moves in the state 2 is \(n_2\) and the number of the down moves in the state 1 is \(n_3\). Then our optimal stopping problems can be rewritten as follows;

\[
\begin{align*}
\bar{v}_0^0(0, 0, 0) &= (K^i - s)^+, \quad i = 1, 2, \\
\bar{v}_n^1(n_1, n_2, n_3) &= \min\{(K^1 - S_n)^+ + \delta_1^n, \max\{(K^1 - S_n)^+, \beta\{p_1q_1\bar{v}_{n+1}^1(n_1 + 1, n_2, n_3) + (1 - p_1)q_1\bar{v}_{n+1}^2(n_1, n_2, n_3 + 1) + (1 - q_1)p_1\bar{v}_{n+1}^2(n_1, n_2, n_3 + 1) + (1 - q_1)(1 - p_1)\bar{v}_{n+1}^2(n_1, n_2, n_3)\}\}, \\
\bar{v}_n^2(n_1, n_2, n_3) &= \min\{(K^2 - S_n)^+ + \delta_2^n, \max\{(K^2 - S_n)^+, \beta\{p_2q_2\bar{v}_{n+1}^2(n_1 + 1, n_2 + 1, n_3) + (1 - p_2)q_2\bar{v}_{n+1}^1(n_1 + 1, n_2, n_3) + (1 - q_2)p_2\bar{v}_{n+1}^1(n_1 + 1, n_2, n_3 + 1) + (1 - q_2)(1 - p_2)\bar{v}_{n+1}^1(n_1 + 1, n_2, n_3)\}\}. \tag{4.7}
\end{align*}
\]

We set the parameters as \(T = 1, r = 0.1, \sigma_1 = 0.1, \sigma_2 = 0.4, p_1 = 0.5, p_2 = 0.8, K^1 = K^2 = 100, \delta_1^n = \delta_1 = 5, \delta_2^n = \delta_2 = 6\) for all \(n\). The optimal exercise regions for the issuer and the investor is represented in Figure 1. Next, we vary a parameter \(S_0, T, \sigma_1, p_1\) or \(\delta_2\) and keep all other parameters fixed. The option value are decreasing in \(S_0\) and increasing in \(T, \sigma_1\) and \(p_1\).

Since the asset price process \(S_n\) defined in equation (4.3) is a martingale, there is no optimal boundary for the investor in \(0 \leq t < T\) for callable American call option (see Remark 3.1). Here, we consider the case that the asset price is not a martingale. We set the probability of the upward jump \(q_1 = q_2 = 0.5\) instead of equation (4.3). In addition, we assume that the \(T = 1, r = 0.1, \sigma_1 = 0.1, \sigma_2 = 0.3, p_1 = 0.5, p_2 = 0.8, K^1 = K^2 = 100, \delta_1 = 5\) and \(\delta_2 = 6\). Figure 2 shows the optimal exercise regions for both players.

5. **Concluding Remarks**

In this paper we consider the discrete time valuation model for callable contingent claims in which the asset price depends on a Markov environment process. The model explicitly incorporates the use of the regime switching. It is shown that such valuation model with the Markov regime switches can be formulated as a coupled optimal stopping problem of a two person game between the issuer and the investor. In particular, we show under some assumptions that there exists a simple optimal call policy for the issuer and optimal exercise policy for the investor which can be described by the control limit values. If the distributions of the state of the economy are stochastically ordered, then we investigate analytical properties of such optimal stopping rules for the issuer and the investor, respectively, possessing a monotone property.
We assume that the asset price follows a random walk with first order stochastic dominance constraint. We wish to extend it to the one with second order stochastic dominance. Moreover, it is of interest to extend it to the three person games among the issuer, investor and the third party like stake holders. If we can directly observe the state of the economy but be able to partially observable, the regime switching model can be formulated as a partially observable Markov one. We shall leave it for future reserch.

References

Figure 2: Optimal exercise boundaries for the callable American call.


