New mechanism of market price observation: liquidity and leptokurtic return distributions

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Abstract

We develop a new mechanism of observing a market price. There exist a hidden price and a corresponding observable market price; the former process is described by a Wiener process, and the observed price is updated and equalized to the hidden price when the change in the hidden process exceeds a threshold. The observed price remains constant unless the change in the hidden process is greater than the threshold. The level of the threshold is proportional to liquidity. A higher level of the threshold is set for a less liquid asset. Then, the observed returns have a leptokurtic distribution. It is shown that the kurtosis of return is proportional to the level of threshold, and that the kurtosis enables us to measure the liquidity. To support our model, we empirically study the spot rate processes estimated from two kinds of Japanese Government Bonds (JGBs), one, more liquid, and the other, less liquid. The empirical results show that the kurtosis of the spot rate processes of the less liquid JGBs are bigger than that of the more liquid JGBs.

Key words: price processes, liquidity, leptokurtic distribution

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1 Introduction

Standard financial theories assume that the return processes follow Gaussian distributions. Examples of their fruitful accomplishments are the CAPM and the option pricing models. On the other hand, it is well known that the return process distributions deviate from the Gaussian distribution, and exhibit leptokurtic properties. To reconcile the theories and the empirical facts, various modifications of return processes have been proposed such as stochastic volatilities, GARCH models, Levy processes, and so forth.

However, few studies pay attention to the price generating process of assets from viewpoint of liquidity. Lesmond, Ogden and Trzcinka (1999) and Chen, Lesmond and Wei (2007) assume that an asset price is described by a hidden return together with a particular observation rule. When the accumulated changes in the hidden return exceed the trading costs, the return becomes observable. They estimate the transaction cost by applying the limited dependent variable (LDV) model. They do not consider any price generating processes. Kagraoka (2005) observes that the yields of illiquid bonds do not change frequently, and follow a jump process. The resultant yield distributions share a leptokurtic property. He proposes to the use of kurtosis as a liquidity measure.

In this paper, we propose a new mechanism pertaining to liquidity and the leptokurtic property of the return distributions. A price process is expressed by a Wiener process with a threshold effect, and the level of the threshold is determined by the liquidity. The resulting return distribution has a leptokurtic feature. It is shown that the kurtosis is proportional to the level of the threshold. We conduct an empirical analysis to support our model. We compare the spot rate processes of two kinds of Japanese Government Bonds (JGBs), with one being more liquid than the other. The empirical results show that the kurtosis of the spot rate processes of the less liquid JGBs is bigger than that of the more liquid JGBs.

The remainder of the paper is organized as follows. In section 2, the return processes are modelled by a Wiener process with a threshold effect, and the relation between liquidity and the kurtosis of return change is shown. In section 3, the empirical evidence supporting our model is presented. Section 4 summarizes the paper and includes discussions on our model.
2 Modelling of the price processes

Dealers specify the bid-ask spreads of the assets by taking into account the liquidity of the assets. They take wide bid-ask spreads for illiquid assets and tight spreads for liquid assets. This is the reason why many researchers take bid-ask spreads as a proxy for liquidity. While dealers constantly change bid and ask prices for liquid assets, they do not frequently change bid and ask prices for illiquid assets. The bid and ask prices of illiquid assets often remain unchanged. Dealers update the bid and ask prices for illiquid assets only when other asset prices change considerably.

We model the price processes of the assets from the viewpoint of liquidity. It is convenient to concentrate on the return processes rather than the price processes. The return processes of the bid (ask) prices are theoretically represented by Wiener processes. On the contrary, the observed return processes are not continuous because of liquidity. Dealers update the bid and ask prices for illiquid assets only when the accumulated change in their returns exceeds the threshold. 1

We formulate the bid (ask) return processes by stochastic processes. We introduce two return processes of the $i$th asset at time $t$, $R^*_{it}$ and $R_{it}$. The former return process is represented by

$$dR^*_{it} = \sigma_i dW_t,$$

where $W_t$ is a Wiener process, and $\sigma_i$ is a volatility function. 2 The stochastic process $R^*_{it}$ is the unobservable ‘true’ return of the $i$th asset at time $t$. The process $R^*_{it}$ would be directly observed under infinite liquidity. The observation rule gives $R_{it}$, and it can be defined as follows. We first define a sequence of stopping times:

$$T_0 = 0,$$

$$T_n = \inf\{ t \mid t > T_{n-1}, \ |R^*_{it} - R_{iT_{n-1}}| > \phi_i \ \} \quad \text{for} \quad n = 1, 2, \ldots.$$ 3

The parameter $\phi_i$ is a threshold for the $i$th asset. The return $R_{it}$ is constructed by the following procedure:

$$R_{it} = \begin{cases} R_{iT_{n-1}} & \text{when} \quad T_{n-1} \leq t < T_n \\ R^*_{it} & \text{when} \quad t = T_n \quad n = 1, 2, \ldots. \end{cases}$$

We give the sample paths of the returns $R^*_{it}$ and $R_{it}$ in Fig. 1. We set $\sigma_i = 0.05$ in eq. (1), and take two thresholds: $\phi_i = 0.15$ and $\phi_i = 0.30$. The greater the

1 The threshold is also closely related to the tick size of the quote on price. The minimum value of the threshold coincides with the tick size.
2 We assume that the riskfree rate is zero for simplicity. It is easy to introduce a non-zero riskfree rate in our model.
threshold \( \phi_i \), the longer the return \( R_{it} \) remains at the same level.

The magnitude of the threshold is proportional to the kurtosis of the distribution of \( R_{it} \). We are going to rely on Monte Carlo simulations to investigate the relations between the threshold, \( \phi_i \), and the kurtosis since we cannot directly monitor the threshold \( \phi_i \). \(^3\) We set \( \sigma_i \) as 0.05, and construct the stochastic processes for \( R_{it} \) with \( \phi_i \) ranging from 0.0025 to 0.3. In Fig. 2, we display the relationship between the kurtosis of the simulated distribution of \( dR_{it} \) and the threshold parameter \( \phi_i \). This relation is clear: the kurtosis increases with \( \phi_i \) but at a faster rate. In Fig. 3, we display the relationship between the standard deviation of the simulated distribution of \( dR_{it} \) and the threshold \( \phi_i \). We find a U-shape relationship between them. We cannot have one-to-one correspondence between the standard deviation and the threshold \( \phi_i \). We take the kurtosis of \( dR_{it} \) as a measure of liquidity in place of \( \phi_i \).

3 Empirical evidence

In the previous section, we present our model and its properties, including that the returns of the less liquid securities have a higher kurtosis. To find empirical evidence, we investigate the spot rate processes of the 10-year and 20-year Japanese Government Bonds (JGBs). The JGBs have been issued by the Government of Japan, and credit risks are common to all JGBs. Thus, the difference between the 10-year and 20-year JGBs is liquidity; the 10-year JGBs are more liquid than the 20-year JGBs. We compare the term structures of the spot rates of the 10-year JGBs and 20-year JGBs for up to ten years, and investigate whether the kurtosis of the spot rate processes of the 20-year JGBs is greater than that of the 10-year JGBs.

We construct the daily spot rate processes of the JGBs as follows. Our data period is from Jan 1, 2005 to December 30, 2009. We estimate the term structures of the spot rates of the 10-year JGBs as well as that of the 20-year JGBs using B-spline (Fisher, Nychka and Zervos (1994)). We set the knot points at zero and at every odd integer up to 21. We do not impose any smoothing condition on the spot rate curves. We have 1,825 daily observations of the term structure of the spot rates. The estimated spot rate curves of the 10-year JGBs and 20-year JGBs are given in Fig. 4 and Fig. 5, respectively. The term structure of the spreads between the spot rates of the 10-year JGBs and that of the 20-year JGBs is depicted in Fig. 6.

It is well known that financial time series are not stationary: volatility clustering is one of the key features. To estimate the kurtosis of the spot rate time

\(^3\) As far as we know, there is no analytical form for the kurtosis of \( dR_{it} \).
series, we employ an extended version of a GARCH model proposed by Léon, Rubio and Serna (2005). Their model allows for time-varying volatility, skewness and kurtosis, and they call it the GARCHSK model. We investigate the spot rate processes, \( r_t(\tau) \), where \( \tau \) stands for term-to-maturity. The stochastic differential equations are given by

\[
dr_t = (\alpha_0 + \alpha_1 r_t) \, dt + \sigma_t \, dZ_t, \tag{5}
\]

where \( Z_t \) is a shock. The Euler-Maruyama discrete time approximation of the continuous time model is

\[
r_t - r_{t-1} = \alpha_0 + \alpha_1 r_{t-1} + \epsilon_t. \tag{6}
\]

The last term is called the shock or residual interchangeably. The variance equation of the shock is of the GARCH(1,1)-type:

\[
\epsilon_t | I_{t-1} \sim (0, h_t), \tag{7}
\]

and

\[
h_t = \beta_0 + \beta_1 \epsilon_{t-1}^2 + \beta_2 h_{t-1}, \tag{8}
\]

where \( I_{t-1} \) is an information set till period \( t - 1 \). The shock is factorized into

\[
\epsilon_t = \sqrt{h_t} \eta_t, \tag{9}
\]

with

\[
E_{t-1}[\eta_t] = 0, \quad E_{t-1}[\eta_t^2] = 1. \tag{10}
\]

The normalized shock, \( \eta_t \), deviates from the normal distribution, and is characterized by

\[
E_{t-1}[\eta_t^3] = s_t, \quad E_{t-1}[\eta_t^4] = k_t. \tag{11}
\]

We consider a restricted version of the GARCHSK model in which the skewness and kurtosis are expressed as

\[
s_t = \gamma_0 + \gamma_1 \eta_{t-1}^3, \tag{12}
\]

and

\[
k_t = \delta_0, \tag{13}
\]

respectively. The model parameters are estimated with the maximum likelihood.

We examine nine spot rates with term-to-maturity 2-10 years. Following this, we estimate nine kurtoses for each spot rate. The JGBs a term-to-maturity less than two years are seldom traded, and the spot rates estimated from these bonds are implausible. The estimated parameters for the 10-year and 20-year JGBs by term-to-maturity are given in Table 1. The kurtosis parameters \( \delta_0 \) are plotted in Fig. 7. The kurtoses are more than three (Gaussian case), and show leptokurtic properties. As shown in the figure, the kurtoses of the spot
rates estimated from the 20-year JGBs are bigger than that estimated from the 10-year JGBs except for maturity at ten years. When we estimate the spot rates from the 10-year JGBs, we need to extrapolate the spot rates at ten years since many 10-year JGBs mature earlier than ten years. Then, ten year spot rates contain estimation errors, and this results in the bigger kurtosis of the 10-year JGBs. We recognize that we need more a comprehensive analysis of the relation between kurtosis and liquidity since the range of the parameter values is contained in the 95% confidence level. Our empirical results show that the spot rates of the less liquid JGBs have bigger kurtoses; this finding is an empirical evidence of our model.

4 Conclusion

In this paper, we model the price processes of the assets from the viewpoint of liquidity. We formulate the return processes by Wiener processes with threshold effects. We numerically show that the threshold is proportional to the kurtosis of the return change distribution. The model is supported by the empirical evidence from the investigation of the spot rates of two kinds of JGBs.

Our hypothetical model should be tested on the price data by brokers. We leave this problem for the future.

References

Table 1
Empirical results of the GARCHSK model.

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<th>$a_1$</th>
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The reported coefficient shown in each row of the table are ML estimates of the GARCHSK model with quasi-maximum likelihood p-values in parenthesis. Spot rates are separately estimated from 10-year and 20-year JGBs. The first column, $\tau$, is term-to-maturity of spot rates. Spot rate is represented by the equation $r_t - r_{t-1} = \alpha_0 + \alpha_1 r_{t-1} + \epsilon_t$ where $\epsilon_t$ is a shock of GARCH-type, $\epsilon_t | h_{t-1} \sim (0, h_t)$ and $h_t = \beta_0 + \beta_1 \epsilon_{t-1}^2 + \beta_2 h_{t-1}$. The shock is decomposed into $\epsilon_t = \sqrt{\eta_1} \eta_t$ with $E_{t-1} [\eta_t^2] = 0$ and $E_{t-1} [\eta_t^4] = 1$. The normalized shock, $\eta_t$, does not follow a normal distribution, and it satisfies $E_{t-1} [\eta_t^2] = \mu_t$ and $E_{t-1} [\eta_t^4] = k_t$. The skewness and the kurtosis are expressed by $\mu_t = \gamma_0 + \gamma_1 \eta_{t-1}^{-1}$, and $k_t = \delta_0$, respectively.

The values in parentheses are $t$-values.
The return $R_{it}^*$ is an unobservable, and it is generated by $dR_{it}^* = \sigma_i dW_t$ where $W_t$ is a Wiener process. The observation rule of $R_{it}^*$ gives $R_{it}$. We define a sequence of stopping times $T_0 = 0$ and $T_n = \inf\{ t \mid t > T_{n-1}, |R_{it}^* - R_{iT_{n-1}}| > \phi_i\}$ for $n = 1, 2, \ldots$. The return $R_{it}$ is constructed by the following procedure,

$$R_{it} = \begin{cases} R_{iT_{n-1}} & \text{when } T_{n-1} \leq t < T_n \\ R_{it}^* & \text{when } t = T_n \quad n = 1, 2, \ldots \end{cases}$$

We set $\sigma_i = 0.05$, and we take two thresholds $\phi_i = 0.15$ and $\phi_i = 0.30$ for comparison. The initial values of $R_{it}^*$ and $R_{it}$ are set equal to 2%.
Fig. 2. Kurtosis of $dR_{it}$ by $\phi_i$.

The return $R^*_t$ is an unobservable, and it is generated by $dR^*_t = \sigma_i dW_t$ where $W_t$ is a Wiener process. The observation rule of $R^*_t$ gives $R_{it}$. We define a sequence of stopping times $T_0 = 0$ and $T_n = \inf \{ t \mid t > T_{n-1}, |R^*_{it} - R_{iT_{n-1}}| > \phi_i \}$ for $n = 1, 2, \ldots$. The return $R_{it}$ is constructed by the following procedure,

$$R_{it} = \begin{cases} R_{iT_{n-1}} & \text{when } T_{n-1} \leq t < T_n \\ R^*_{it} & \text{when } t = T_n \quad n = 1, 2, \ldots \end{cases}$$

We set $\sigma_i$ to 0.05. We generate a sample path of $R_{it}$ with $t = 1, 2, \ldots, 1000$, and calculate the kurtosis of the sample path. We conduct Monte Carlo simulations 1000 times, and take the average of the kurtosis. The threshold varies from 0.0025 to 0.3.
Fig. 3. Standard deviation by $\phi_i$.

The return $R^*_t$ is an unobservable, and it is generated by $dR^*_t = \sigma_i \, dW_t$ where $W_t$ is a Wiener process. The observation rule of $R^*_t$ gives $R_t$. We define a sequence of stopping times $T_0 = 0$ and $T_n = \inf \{ t \mid T_{n-1} < t, \ \left| R^*_t - R^*_n \right| > \phi_i \}$ for $n = 1, 2, \ldots$. The return $R_t$ is constructed by the following procedure,

$$R_t = \begin{cases} R^*_{tT_{n-1}} & \text{when } T_{n-1} \leq t < T_n \\ R^*_t & \text{when } t = T_n \quad n = 1, 2, \ldots \end{cases}$$

We set $\sigma_i$ to 0.05. We generate a sample path of $R_t$ with $t = 1, 2, \ldots, 1000$, and we calculate standard deviation of the sample path. We conduct Monte Carlo simulations 1000 times, and take the average of the standard deviations. The threshold varies from 0.0025 to 0.3.
Fig. 4. Term structure of spot rate estimated from 10-year JGBs. Term structure of spot rate of the 10-year JGBs is estimated by B-spline.
Fig. 5. Term structure of spot rate estimated from 20-year JGBs. Term structure of spot rate of the 20-year JGBs is estimated by B-spline.
Fig. 6. Term structure of the spread of spot rates between the 10-year and the 20-year JGBs. Term structure of the spread of spot rates is depicted.
Fig. 7. Kurtosis parameter $\delta_0$.
Kurtosis parameters of the 10-year JGBs (red line) and that of the 20-year JGBs (blue line) are plotted with 95% confidence intervals (dotted line).