Leptokurtic properties of JGB yield processes

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Abstract


\[ y_t - y_{t-1} = \alpha_0 + \alpha_1 y_{t-1} + \epsilon_t \]

where \( \epsilon_t \) is a shock of a GARCH-type, \( \epsilon_t \sim (0, h_t) \) and \( h_t = \beta_0 + \beta_1 \epsilon_{t-1}^2 + \beta_2 h_{t-1} \). The shock is factorized into \( \epsilon_t = \sqrt{h_t} \eta_t \) with \( E_{t-1}[\eta_t] = 0 \) and \( E_{t-1}[\eta_t^2] = 1 \). The normalized shock, \( \eta_t \), does not follow a normal distribution, and it satisfies \( E_{t-1}[\eta_t^3] = s_t \) and \( E_{t-1}[\eta_t^4] = k_t \). The skewness and the kurtosis are expressed by

\[ s_t = \gamma_0 + \gamma_1 \eta_{t-1}^3 + \gamma_2 s_{t-1}, \]

and

\[ k_t = \delta_0 + \delta_1 \eta_{t-1}^4 + \delta_2 k_{t-1}, \]

respectively. The model parameters are estimated with the maximum likelihood. The model is applied to daily yields of JGBs from January 2000 to December 2004. Empirical results unveil the following new findings; (i) kurtosis are significant and comparable for all JGBs (in the range of 3.387-3.445), and the distributions of yield changes have leptokurtic properties; (ii) skewnesses are weak for all JGBs; (iii) volatilities are comparable for all JGBs.

Key words: yield processes, GARCH, leptokurtic

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Preprint submitted to Elsevier 5 February 2010
1 Introduction

Many researchers have investigated stochastic processes of the interest rates. They have studied mainly the short rates, however, no consensus has been reached so far on the parametric specification of the drift and diffusion function of the short rate processes. Most models assume that the short rates follow Wiener processes; the Vasicek and Hull-White models are well known among them. Chan, Karolyi, Longstaff and Sanders (1992) (CKLS) compare a variety of parametric short rate models expressed by

\[ dr_t = (\alpha_0 + \alpha_1 r_t) dt + \psi r_t^\kappa dW_t, \]  

(1)

where \( dW_t \) is a Wiener process. The Euler-Maruyama discrete time approximation of the continuous time model gives

\[ r_t - r_{t-1} = \alpha_0 + \alpha_1 r_{t-1} + \epsilon_t, \]  

(2)

with

\[ \epsilon_t | I_{t-1} \sim N(0, \psi^2 r_t^{2\kappa}), \]  

(3)

where \( I_{t-1} \) is the information set at time \( t-1 \). They conduct a point estimates of the parameters and obtain parameter value \( \kappa \) of 1.5. This means that the volatility increases with the level of the interest rates, the so-called level effect. They conclude that the dependence of the volatility on the level of the short rate is one of the most important features. Thus none of the traditional short rate models is consistent with the result of CKLS.

The GARCH model is a popular approach to treat time series of the interest rates since the GARCH model can fit to the volatility clustering. Brenner, Harjes and Kroner (1996) is the first paper which extends CKLS by assuming that \( \epsilon_t \) follows a GARCH process. Their empirical results show that the presence of rather extreme conditional heteroscedastic volatility effects in the interest rate dynamics, but with a weaker level effect with \( \kappa = 0.5 \). There are several papers which apply various GARCH models to short rates, however, there is no consensus on the intensity of the level effect (Koedijk, Nissen, Schotman, and Wolff (1997), Andersen and Lund (1997), Ball and Torous (1999), and Bali (2000)).

The goal of our study is to uncover the distribution of the shock in eq. (2) in the Japanese market. We pay attention to the leptokurtic property of the shock. The keptokurtic property of asset return distribution is very important in risk management (Guermat and Harris (2002)) and also derivative pricing (Das and Sundaram (1999)). We expect that more flexible description of the shock does not need the interest rate dependence on its level. We examine the shock on yield changes without any level-dependence of volatility, while we
respect the GARCH-effect. Léon, Rubio and Serna (2005) propose a GARCH-type model allowing for time-varying volatility, skewness and kurtosis. They estimate the model parameters assuming a Gram-Charlier series expansion of the normal density function for the shock. They apply the model to daily returns of stock indices and exchange rates, and find significant presence of conditional skewness and kurtosis. They confirm that specifications allowing for time-varying skewness and kurtosis outperform specifications with constant third and forth moments.

In contrast to previous studies on investigating dynamics of short rates, we investigate long-term yields, especially yield processes of the Japanese Government Bonds (JGBs). When changes of the short rates follow normal distributions, the resultant yield processes approximately follow normal distributions. Extremely low interest rate policy of the central bank compels the short rates stick to almost zero, and the short rates are constant. The CD 3 month rates nor Treasury Bill rates are not considered to be proper references to the short rates. Most studies have examined the short rate processes, and inferred the long rate processes. On contrary, we conduct time series analysis of the yield processes of the JGBs, and we infer the short rate processes.

The rest of the paper is organized as follows. Section 2 presents our model. Data and empirical results are given in Section 3. Section 4 concludes.
2 Model

We employ an extended version of a GARCH model proposed by Léon, Rubio and Serna (2005). Their model allows for time-varying volatility, skewness and kurtosis, and they call it GARCHSK model. We give a brief summary of their model to make the paper self-contained.

We consider a yield process, \( y_t \),

\[
dy_t = (\alpha_0 + \alpha_1 y_t) \, dt + \sigma_t \, dZ_t, \tag{4}
\]

where \( Z_t \) is a shock. The Euler-Maruyama discrete time approximation of the continuous time model is

\[
y_t - y_{t-1} = \alpha_0 + \alpha_1 y_{t-1} + \epsilon_t. \tag{5}
\]

The last term is called the shock or yield residual interchangeably. The variance equation of the shock is a GARCH(1,1)-type,

\[
\epsilon_t | I_{t-1} \sim (0, h_t), \tag{6}
\]

and

\[
h_t = \beta_0 + \beta_1 \epsilon_{t-1}^2 + \beta_2 h_{t-1}, \tag{7}
\]

where \( I_{t-1} \) is an information set till period \( t - 1 \). The shock is factorized into

\[
\epsilon_t = \sqrt{h_t} \eta_t, \tag{8}
\]

with

\[
E_{t-1}[\eta_t] = 0, \quad E_{t-1}[\eta_t^2] = 1. \tag{9}
\]

The normalized shock, \( \eta_t \), deviates from a normal distribution, and it is characterized by

\[
E_{t-1}[\eta_t^3] = s_t, \quad E_{t-1}[\eta_t^4] = k_t. \tag{10}
\]

The skewness and kurtosis are expressed as

\[
s_t = \gamma_0 + \gamma_1 \eta_{t-1}^3 + \gamma_2 s_{t-1}, \tag{11}
\]

and

\[
k_t = \delta_0 + \delta_1 \eta_{t-1}^4 + \delta_2 k_{t-1}, \tag{12}
\]

respectively. A Gram-Charlier series expansion of the density function of the normalized shock, \( \eta_t \), conditional on \( I_{t-1} \) is

\[
g(\eta_t | I_{t-1}) = \phi(\eta_t) \left( 1 + \frac{s_t}{3!} (\eta_t^3 - 3\eta_t) + \frac{k_t - 3}{4!} (\eta_t^4 - 6\eta_t^2 + 3) \right) \tag{13}
\]

\[
=: \phi_t(\eta_t) \Psi(\eta_t), \tag{14}
\]
where \( \phi(\bullet) \) denotes the density function for the standard normal distribution. To guarantee both the positivity and normalization of the density function, they modify the density function of the normalized shock as
\[
f(\eta_t|I_{t-1}) = \frac{\phi(\eta_t) \Psi(\eta_t)^2}{\Gamma_t},
\]
where
\[
\Gamma_t = 1 + \frac{s_t}{3!} + \frac{k_t - 3}{4!} .
\]
The conditional distribution of \( \epsilon_t = \sqrt{h_t} \eta_t \) has probability density function \( h_t^{-1/2} f(\eta_t|I_{t-1}) \). The log-likelihood function is
\[
\ell_t = -\frac{1}{2} \ln h_t - \frac{1}{2} \eta_t^2 + \ln(\Psi(\eta_t))^2 - \ln \Gamma_t .
\]
The model parameters are estimated with the maximum likelihood.

As a benchmark, we choose a GARCH(1,1) model (hereafter we simply call it GARCH or GARCHSK0) in which the shock \( \epsilon_t \) follows a standard normal distribution,
\[
\epsilon_t|I_{t-1} \sim N(0, h_t) .
\]
3 Empirical results

Data are provided by the Japan Securities Dealers Association (JSDA). Our dataset records daily yields of 10-year Japanese Government Bonds (JGBs) from January 4, 2000 to December 31, 2004. JGBs are the most liquid bonds in the Japanese bond market, and they work as benchmarks for bond investments. We select the JGBs which have no missing yield in the data period since the GARCH-type models require long time series of data in their parameter estimation. Finally we have 1243 daily observations on 22 JGB yields; issue date covers from January 27, 1997 to June 21, 1999, and coupon rates are between 0.9% and 2.7%. The oldest issue is the 192nd JGB, issued on January 27, 1997, and it is redeemed on March 20, 2007. The latest issue is the 213th JGB, issued on June 21, 1999, and it is redeemed on June 22, 2009. As we explain later, empirical results are comparable across the 22 JGBs, and we report the results on four of them because of a space limitation; issue dates of these bonds are at every eight to eleven months. Their attributes and summary statistics of the yield residual, $\epsilon_t$ in eq. (5), are given in Table 1 and Table 2, respectively. Yields trajectories of the four JGBs are depicted in Fig. 1. Time series of yield and yield residual are shown in Fig. 2–Fig. 5.

We test three GARCHSK models as well as the GARCH model. As shown below, the result of the GARCHSK model shows that only $\gamma_1$ and $\delta_0$ are statistically significant in the skewness and kurtosis equations. We introduce two reduced GARCHSK models, GARCHSK2 and GARCHSK3 models. The GARCHSK2 model has two free parameters, $\gamma_0$ and $\delta_0$, in the skewness and kurtosis specifications; rest of the parameters, $\gamma_1$, $\gamma_2$, $\delta_1$, and $\delta_2$, are set to zero. The GARCHSK3 model has three free parameters in the skewness and kurtosis specifications; rest of the parameters, $\gamma_2$, $\delta_1$, and $\delta_2$, are set to zero. To distinguish the original GARCHSK model from two variants, we name it as GARCHSK6 model; the GARCHSK6 model is a full model, and there are six free parameters in the skewness and kurtosis specifications. Similarly, we call the GARCH model as GARCHSK0 model.

As given in eq. (17), the log-likelihood function is highly nonlinear in the parameters. We have to be very careful to choose the starting values of the parameters in estimation procedure. To estimate the parameters, we proceed in five steps. We first conduct the simple regression of eq. (5). At the second step, we estimate the GARCHSK0 model with the starting values which are estimated in the simple regression. Then we estimate the GARCHSK2 models with the starting values which are obtained in the GARCHSK0 model. The estimated parameters in the GARCHSK2 model are taken as the starting points for the GARCHSK3 model. Finally, the output of the GARCHSK3 models are used as the starting points for the GARCHSK6 model. The results of the GARCHSK0, GARCHSK2, GARCHSK3, and GARCHSK6 models are given
in Table 3–Table 6, respectively. The estimated parameters for the all JGBs are depicted in Fig. 6–Fig. 9. All parameters are comparable across the JGBs.

We first examine the result of the GARCHSK0 model. The parameters of the mean equation, \( \alpha_0 \) and \( \alpha_1 \) are not statistically significant. In the data period, yields decline from 1.6% to 0.3% on average, and the parameter \( \alpha_0 \) takes negative value. The parameters of the variance equation, \( \beta_0 \), \( \beta_1 \), and \( \beta_2 \), are statistically significant. The constant \( \beta_0 \) becomes the greater from 0.0545 to 0.1873 as the remaining term-to-maturity becomes the longer. This fact is reasonable since the term structure of the yield is upward-sloping with respect to remaining term-to-maturity, and long-term yields are more volatile than short-term ones. The remaining parameters, \( \beta_1 \) and \( \beta_2 \), take around 0.170 and 0.828, respectively. The sum \( \beta_1 + \beta_2 \) is very close to one, and this fact implies that the effect of a yield shock on current volatility declines very slowly.

Next we examine the result of the GARCHSK2 model. The parameters of the mean equation, \( \alpha_0 \) and \( \alpha_1 \) are not statistically significant. The parameters of the variance equation are slightly different from that of the GARCHSK0 model. The constant \( \beta_0 \) becomes the greater from 0.0312 to 0.1289 as the remaining term-to-maturity becomes the longer. The remaining parameters, \( \beta_1 \) and \( \beta_2 \), take around 0.146 and 0.851, respectively. The parameter of the skewness equation, \( \gamma_0 \), takes comparable values around 0.0618. We can not reject a null hypothesis \( \gamma_0 = 0 \) at the 95% confidence level. The parameter of the kurtosis equation is statistically significant, and it is greater than 3 which corresponds to the normal distribution. The constant \( \delta_0 \) becomes the smaller from 3.4920 to 3.4227 as the remaining term-to-maturity becomes the longer. This means that short-term yields are more leptokurtic than long-term yields, however this fact is not so conclusive since the range of the parameter values are contained in the 95% confidence level.

Subsequently, we examine the result of the GARCHSK3 model. All of the estimated parameters are similar to that of the GARCHSK2 model. The parameters of the mean equation, \( \alpha_0 \) and \( \alpha_1 \) are not statistically significant. The constant \( \beta_0 \) becomes the greater from 0.0362 to 0.1218 as the remaining term-to-maturity becomes the longer. The remaining parameters, \( \beta_1 \) and \( \beta_2 \), take around 0.164 and 0.833, respectively. The parameters of the skewness equation, \( \gamma_0 \) and \( \gamma_1 \), take comparable values around 0.0342 and 0.0314, respectively. We can not reject the null hypothesis \( \gamma_0 = 0 \) at the 95% confidence level. The another parameter, \( \gamma_1 \), is statistically significant at the 99% confidence level. It means that the distribution of the yield residual is right skewed, however, its effect on skewness is weak since it is the coefficient to \( \eta^3 \). We conclude that the skewness is very weak in the yield residuals. The parameter of the kurtosis equation is statistically significant and it is greater than 3. The constant \( \delta_0 \) becomes the smaller from 3.445 to 3.387 as the remaining term-to-maturity becomes the longer.
Finally, we examine the result of the GARCHSK6 model. The estimated parameters of the kurtosis equation are very different from that of the GARCHSK3 model. We suspect that the model is not well-specified.

We conduct the likelihood ratio (LR) test to compare the GARCHSK models. Results are given in Table 7. The value of the LR statistic is quite large for the GARCHSK2, GARCHSK3, and GARCHSK6 models compared with that of the GARCHSK0 model, indicating the rejection of the null hypothesis that the true distribution of the shock is the standard normal distribution. We convince that the GARCHSK2, GARCHSK3, and GARCHSK6 models remarkably overwhelm the GARCHSK0 model. The GARCHSK3 model is best model among the GARCHSK models regarding the LR statistics.
4 Conclusion

We investigate the JGB yield processes by applying the GARCHSK models allowing for time-varying volatility, skewness and kurtosis. The model is applied to the daily data of the JGB yields, and the parameters are estimated with the maximum likelihood. Empirical results unveil the following new findings;

- The GARCHSK models outperforms the GARCHSK0 model.
- The GARCHSK3 model is the best model in the GARCHSK models based on the LR tests.
- Kurtosis are significant and comparable for all JGBs (in the range of 3.387-3.445 in the GARCHSK3 model), and the distribution of yield changes have leptokurtic properties.
- Skewness are weak for all JGBs.
- Volatilities are comparable for all JGBs.

Litterman and Scheinkman (1991) report that the variation of term structure of interest rates is explained almost by three factors; called level, steepness and curvature. The level factor causes parallel shifts of term structure of interest rates, and it amount to 90%. Then variations of JGB yields are commonly generated by the level factor. This is the reason why the estimated parameters are comparable across the JGBs.

In low interest rate environment such as in Japan, yield levels are close to zero, and tick sizes of quote are rough compared to the yield levels. Then, changes in the yields might be represented by jump processes rather than continuous processes.

We conjecture that the short-term yields are more leptokurtic than the long-term yields. The short-term bonds are less liquid than the long-term bonds, and the short-term yields do not follow continuous processes. The short-term yields are lower than the long-term yields, and ratios of the tick size of quote to the yield level are larger for the short-term bonds than the long-term bonds.

We incorporate the level effect in the GARCHSK model,

\[ \epsilon_t = \sqrt{h_t} \eta_t, \]
\[ \epsilon_t|I_{t-1} \sim (0, r^{2\kappa}_{t-1} h_t), \]

where \( \eta_t \) has skewness and kurtosis. We conduct parameter estimation with the maximum likelihood, however, we do not succeed to estimate parameters because behavior of the likelihood function is not appropriate. We think that the level effect and an extension of the GARCH with skewness and kurtosis are incompatible.
Our results imply that yield changes have stable leptokurtic distributions. This fact suggests that risk measures, such as VaR or expected short fall, should be estimated with the kurtosis consideration. We can apply the GARCHSK model to evaluate how important the kurtosis of distribution is.
References


Table 1
JGB attribute.

<table>
<thead>
<tr>
<th>issue no.</th>
<th>coupon</th>
<th>issue amount</th>
<th>issue date</th>
<th>maturity date</th>
</tr>
</thead>
<tbody>
<tr>
<td>192</td>
<td>2.7</td>
<td>1309</td>
<td>Jan. 27, 1997</td>
<td>Mar. 20, 2007</td>
</tr>
<tr>
<td>200</td>
<td>2.0</td>
<td>6059</td>
<td>Nov. 20, 1997</td>
<td>Dec. 20, 2007</td>
</tr>
<tr>
<td>207</td>
<td>0.9</td>
<td>5217</td>
<td>Oct. 20, 1998</td>
<td>Dec. 22, 2008</td>
</tr>
<tr>
<td>213</td>
<td>1.4</td>
<td>1640</td>
<td>June 21, 1999</td>
<td>June 22, 2009</td>
</tr>
</tbody>
</table>

Bond attributes of JGB is summarized. Issue amount is in billion yen.

Table 2
Summary statistics of yield residuals.

<table>
<thead>
<tr>
<th>issue no.</th>
<th>mean</th>
<th>median</th>
<th>max</th>
<th>min</th>
<th>std</th>
<th>skewness</th>
<th>kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>192</td>
<td>$-9.8994E-17$</td>
<td>0.0072</td>
<td>13.0276</td>
<td>$-11.7971$</td>
<td>2.2595</td>
<td>0.255</td>
<td>6.873</td>
</tr>
<tr>
<td>200</td>
<td>$-1.5782E-16$</td>
<td>$-0.0014$</td>
<td>15.0589</td>
<td>$-11.9221$</td>
<td>2.5328</td>
<td>0.267</td>
<td>6.516</td>
</tr>
<tr>
<td>207</td>
<td>$-8.1479E-17$</td>
<td>$-0.0362$</td>
<td>15.3846</td>
<td>$-11.8250$</td>
<td>2.7228</td>
<td>0.364</td>
<td>6.640</td>
</tr>
<tr>
<td>213</td>
<td>$3.3510E-16$</td>
<td>$-0.0572$</td>
<td>18.0028</td>
<td>$-12.1227$</td>
<td>2.8683</td>
<td>0.472</td>
<td>7.074</td>
</tr>
</tbody>
</table>

Yield changes are represented by the equation \( y_t - y_{t-1} = \alpha_0 + \alpha_1 y_{t-1} + \epsilon_t \), and we call \( \epsilon_t \) as yield residual. Yields and the yield residuals are measured in basis point (bp).
<table>
<thead>
<tr>
<th>issue no.</th>
<th>mean equation</th>
<th>variance equation</th>
<th>skewness equation</th>
<th>kurtosis equation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \alpha_0 )</td>
<td>( \alpha_1 )</td>
<td>( \beta_0 )</td>
<td>( \beta_1 )</td>
</tr>
<tr>
<td>192</td>
<td>-0.0770</td>
<td>-0.0023</td>
<td>0.0545</td>
<td>0.1821</td>
</tr>
<tr>
<td></td>
<td>(0.1427)</td>
<td>(0.0501)</td>
<td>(0.0001)</td>
<td>(0.0000)</td>
</tr>
<tr>
<td>200</td>
<td>-0.0845</td>
<td>-0.0020</td>
<td>0.0861</td>
<td>0.1675</td>
</tr>
<tr>
<td></td>
<td>(0.2645)</td>
<td>(0.1197)</td>
<td>(0.0003)</td>
<td>(0.0000)</td>
</tr>
<tr>
<td>207</td>
<td>-0.1204</td>
<td>-0.0012</td>
<td>0.1461</td>
<td>0.1677</td>
</tr>
<tr>
<td></td>
<td>(0.2287)</td>
<td>(0.2853)</td>
<td>(0.0019)</td>
<td>(0.0000)</td>
</tr>
<tr>
<td>213</td>
<td>-0.1533</td>
<td>-0.0007</td>
<td>0.1873</td>
<td>0.1631</td>
</tr>
<tr>
<td></td>
<td>(0.2040)</td>
<td>(0.5416)</td>
<td>(0.0016)</td>
<td>(0.0000)</td>
</tr>
</tbody>
</table>

The reported coefficient shown in each row of the table are ML estimates of the GARCHSK0 model with quasi-maximum likelihood p-values in parenthesis. A yield change is represented by the equation \( y_t - y_{t-1} = \alpha_0 + \alpha_1 y_{t-1} + \epsilon_t \) where \( \epsilon_t \) is a shock of GARCH-type, \( \epsilon_t | h_{t-1} \sim (0, h_t) \) and \( h_t = \beta_0 + \beta_1 \epsilon_{t-1}^2 + \beta_2 h_{t-1} \). The shock is decomposed into \( \epsilon_t = \sqrt{h_t} \eta_t \) with \( E_{t-1}[\eta_t] = 0 \) and \( E_{t-1}[\eta_t^2] = 1 \). The normalized shock, \( \eta_t \), follows a normal distribution, and it satisfies \( E_{t-1}[\eta_t^3] = s_t = 0 \) and \( E_{t-1}[\eta_t^4] = k_t = 3 \).
Table 4
Empirical results of the GARCHSK2 model.

<table>
<thead>
<tr>
<th>issue no.</th>
<th>mean equation</th>
<th>variance equation</th>
<th>skewness equation</th>
<th>kurtosis equation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td>$\alpha_1$</td>
<td>$\beta_0$</td>
<td>$\beta_1$</td>
</tr>
<tr>
<td>192</td>
<td>-0.0773</td>
<td>-0.0015</td>
<td>0.0312</td>
<td>0.1558</td>
</tr>
<tr>
<td></td>
<td>(0.1166)</td>
<td>(0.2125)</td>
<td>(0.0048)</td>
<td>(0.0000)</td>
</tr>
<tr>
<td>207</td>
<td>-0.0955</td>
<td>-0.0010</td>
<td>0.0429</td>
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<tr>
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<td>(0.1661)</td>
<td>(0.4062)</td>
<td>(0.0146)</td>
<td>(0.0000)</td>
</tr>
<tr>
<td>213</td>
<td>-0.1453</td>
<td>-0.0004</td>
<td>0.0808</td>
<td>0.1471</td>
</tr>
<tr>
<td></td>
<td>(0.1164)</td>
<td>(0.6855)</td>
<td>(0.0491)</td>
<td>(0.0000)</td>
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<tr>
<td></td>
<td>(0.0987)</td>
<td>(0.9763)</td>
<td>(0.0340)</td>
<td>(0.0000)</td>
</tr>
</tbody>
</table>

The reported coefficient shown in each row of the table are ML estimates of the GARCHSK2 model with quasi-maximum likelihood p-values in parenthesis. A yield change is represented by the equation $y_t - y_{t-1} = \alpha_0 + \alpha_1 y_{t-1} + \epsilon_t$ where $\epsilon_t$ is a shock of GARCH-type, $\epsilon_t | h_{t-1} \sim (0, h_t)$ and $h_t = \beta_0 + \beta_1 \epsilon_{t-1}^2 + \beta_2 h_{t-1}$. The shock is decomposed into $\epsilon_t = \sqrt{h_t} \eta_t$ with $E_{t-1}[\eta_t] = 0$ and $E_{t-1}[\eta_t^2] = 1$. The normalized shock, $\eta_t$, does not follow a normal distribution, and it satisfies $E_{t-1}[\eta_t^3] = s_t$ and $E_{t-1}[\eta_t^4] = k_t$. The skewness and the kurtosis are expressed by $s_t = \gamma_0$, and $k_t = \delta_0$, respectively.
Table 5
Empirical results of the GARCHSK3 model.

<table>
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<tr>
<th>issue no.</th>
<th>mean equation</th>
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<th>skewness equation</th>
<th>kurtosis equation</th>
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<td></td>
<td>(0.0298)</td>
<td>(0.1476)</td>
<td>(0.0002)</td>
<td>(0.0000)</td>
</tr>
<tr>
<td>200</td>
<td>-0.1400</td>
<td>-0.0010</td>
<td>0.0541</td>
<td>0.1745</td>
</tr>
<tr>
<td></td>
<td>(0.0287)</td>
<td>(0.3711)</td>
<td>(0.0012)</td>
<td>(0.0000)</td>
</tr>
<tr>
<td>207</td>
<td>-0.1876</td>
<td>-0.0003</td>
<td>0.0840</td>
<td>0.1532</td>
</tr>
<tr>
<td></td>
<td>(0.0303)</td>
<td>(0.7760)</td>
<td>(0.0262)</td>
<td>(0.0000)</td>
</tr>
<tr>
<td>213</td>
<td>-0.2359</td>
<td>0.0002</td>
<td>0.1218</td>
<td>0.1465</td>
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<tr>
<td></td>
<td>(0.0257)</td>
<td>(0.8372)</td>
<td>(0.0259)</td>
<td>(0.0000)</td>
</tr>
</tbody>
</table>

The reported coefficient shown in each row of the table are ML estimates of the GARCHSK3 model with quasi-maximum likelihood p-values in parenthesis. A yield change is represented by the equation $y_t - y_{t-1} = \alpha_0 + \alpha_1 y_{t-1} + \epsilon_t$ where $\epsilon_t$ is a shock of GARCH-type, $\epsilon_t | I_{t-1} \sim (0, h_t)$ and $h_t = \beta_0 + \beta_1 \epsilon_{t-1}^2 + \beta_2 h_{t-1}$. The shock is decomposed into $\epsilon_t = \sqrt{h_t} \eta_t$ with $E_{t-1}[\eta_t] = 0$ and $E_{t-1}[\eta_t^2] = 1$. The normalized shock, $\eta_t$, does not follow a normal distribution, and it satisfies $E_{t-1}[\eta_t^3] = s_t$ and $E_{t-1}[\eta_t^4] = k_t$. The skewness and the kurtosis are expressed by $s_t = \gamma_0 + \gamma_1 \eta_{t-1}^3$, and $k_t = \delta_0$, respectively.
Table 6
Empirical results of the GARCHSK6 model.

<table>
<thead>
<tr>
<th>issue no.</th>
<th>mean equation</th>
<th>variance equation</th>
<th>skewness equation</th>
<th>kurtosis equation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\alpha_0$ $\alpha_1$</td>
<td>$\beta_0$ $\beta_1$ $\beta_2$</td>
<td>$\gamma_0$ $\gamma_1$ $\gamma_2$</td>
<td>$\delta_0$ $\delta_1$ $\delta_2$</td>
</tr>
<tr>
<td>192</td>
<td>-0.0961 -0.0017</td>
<td>0.0354 0.1783 0.8207</td>
<td>0.0388 0.0312 -0.0348</td>
<td>3.3490 -0.0024 0.0308</td>
</tr>
<tr>
<td></td>
<td>(0.0400) (0.1304)</td>
<td>(0.0003) (0.0000) (0.0000)</td>
<td>(0.3323) (0.0001) (0.6918)</td>
<td>(0.0005) (0.3746) (0.9114)</td>
</tr>
<tr>
<td>200</td>
<td>-0.1326 -0.0011</td>
<td>0.0527 0.1698 0.8292</td>
<td>0.0286 0.0316 -0.0288</td>
<td>3.2524 -0.0028 0.0531</td>
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<tr>
<td></td>
<td>(0.0392) (0.3307)</td>
<td>(0.0012) (0.0000) (0.0000)</td>
<td>(0.4619) (0.0003) (0.7502)</td>
<td>(0.0000) (0.3312) (0.8158)</td>
</tr>
<tr>
<td>207</td>
<td>-0.1851 -0.0003</td>
<td>0.0834 0.1510 0.8458</td>
<td>0.0391 0.0302 -0.0545</td>
<td>3.0612 -0.0010 0.0981</td>
</tr>
<tr>
<td></td>
<td>(0.0332) (0.7617)</td>
<td>(0.0261) (0.0000) (0.0000)</td>
<td>(0.3296) (0.0005) (0.5901)</td>
<td>(0.0091) (0.7248) (0.7764)</td>
</tr>
<tr>
<td>213</td>
<td>-0.2335 0.0002</td>
<td>0.1210 0.1444 0.8465</td>
<td>0.0525 0.0301 -0.0598</td>
<td>2.9074 -0.0009 0.1425</td>
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<td></td>
<td>(0.0277) (0.8486)</td>
<td>(0.0258) (0.0000) (0.0000)</td>
<td>(0.1938) (0.0005) (0.5311)</td>
<td>(0.0193) (0.7491) (0.6968)</td>
</tr>
</tbody>
</table>

The reported coefficient shown in each row of the table are ML estimates of the GARCHSK6 model with quasi-maximum likelihood $p$-values in parenthesis. A yield change is represented by the equation $y_t - y_{t-1} = \alpha_0 + \alpha_1 y_{t-1} + \epsilon_t$ where $\epsilon_t$ is a shock of GARCH-type, $\epsilon_t|t-1 \sim (0, h_t)$ and $h_t = \beta_0 + \beta_1 \epsilon_{t-1}^2 + \beta_2 h_{t-1}$. The shock is decomposed into $\epsilon_t = \sqrt{h_t} \eta_t$ with $E_t[\eta_t] = 0$ and $E_t[\eta_t^2] = 1$. The normalized shock, $\eta_t$, does not follow a normal distribution, and it satisfies $E_{t-1}[\eta_t^2] = \sigma_t$ and $E_{t-1}[\eta_t^4] = \kappa_t$. The skewness and the kurtosis are expressed by $\gamma_t = \gamma_0 + \gamma_1 \eta_{t-1}^3 + \gamma_2 \eta_{t-1}$, and $k_t = \delta_0 + \delta_1 \eta_{t-1}^4 + \delta_2 k_{t-1}$, respectively.
<table>
<thead>
<tr>
<th>issue no.</th>
<th>log likelihood</th>
<th>LR</th>
</tr>
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<td>GARCHSK2</td>
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<tr>
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<td>-2462.82</td>
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<td>-2722.97</td>
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<td>207</td>
<td>-2857.36</td>
<td>-2829.73</td>
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<tr>
<td>213</td>
<td>-2930.37</td>
<td>-2902.96</td>
</tr>
</tbody>
</table>

This table shows the value of the maximum log-likelihood function ($\log L$) and the likelihood ratio (LR). A yield change is represented by the equation $y_t - y_{t-1} = \alpha_0 + \alpha_1 y_{t-1} + \epsilon_t$ where $\epsilon_t$ is a shock of GARCH-type, $\epsilon_t | I_{t-1} \sim (0, h_t)$ and $h_t = \beta_0 + \beta_1 \epsilon_{t-1}^2 + \beta_2 h_{t-1}$. The shock is decomposed into $\epsilon_t = \sqrt{h_t} \eta_t$ with $E_{t-1}[\eta_t] = 0$ and $E_{t-1}[\eta_t^2] = 1$. The normalized shock, $\eta_t$, does not follow a normal distribution, and it satisfies $E_{t-1}[\eta_t^2] = s_t$ and $E_{t-1}[\eta_t^4] = k_t$. The skewness and the kurtosis are expressed by $s_t = \gamma_0 + \gamma_1 \eta_{t-1}^3 + \gamma_2 \eta_{t-1}$, and $k_t = \delta_0 + \delta_1 \eta_{t-1}^0 + \delta_2 k_{t-1}$, respectively. \text{GARCHSK0} assumes that $\gamma_0, \gamma_1, \gamma_2, \delta_1$, and $\delta_2$ are zero and $\delta_0$ is three. \text{GARCHSK2} assumes that $\gamma_1, \gamma_2, \delta_1$, and $\delta_2$ are zero. \text{GARCHSK3} assumes that $\gamma_2, \delta_1$, and $\delta_2$ are zero. \text{GARCHSK6} have no assumption on the parameters in the skewness and kurtosis equations. LR1, LR2, and LR3 are asymptotic $p$-value of the likelihood ratio of \text{GARCHSK3} to \text{GARCHSK0}, \text{GARCHSK3} to \text{GARCHSK2}, and \text{GARCHSK6} to \text{GARCHSK3}, respectively.
Fig. 1. The yield processes of JGBs.

Yield processes of the JGBs.
Fig. 2. Times series of the yield and the yield residual of JGB 192nd.
Fig. 3. Times series of the yield and the yield residual of JGB 200th.
Fig. 4. Times series of the yield and the yield residual of JGB 207nd.
Fig. 5. Times series of the yield and the yield residual of JGB 213th.
Fig. 6. The estimated parameters of the GARCHSK0 model.

The X-axis is the bond issue number.
Fig. 7. The estimated parameters of the GARCHSK2 model.

The X-axis is the bond issue number.
GARCHSK3 : mean equation: $\alpha_0$

GARCHSK3 : variance equation: $\beta_0$

GARCHSK3 : mean equation: $\alpha_1$

GARCHSK3 : variance equation: $\beta_1$

GARCHSK3 : variance equation: $\beta_2$

GARCHSK3 : skewness equation: $\gamma_0$

GARCHSK3 : skewness equation: $\gamma_1$

GARCHSK3 : skewness equation: $\gamma_2$

GARCHSK3 : kurtosis equation: $\delta_0$

GARCHSK3 : kurtosis equation: $\delta_1$

GARCHSK3 : kurtosis equation: $\delta_2$

Fig. 8. The estimated parameters of the GARCHSK3 model.

The X-axis is the bond issue number.
Fig. 9. The estimated parameters of the GARCHSK6 model.

The X-axis is the bond issue number.